LOYOLA COLLEGE (AUTONOMOUS) CHENNAI – 600 034

M.Sc. DEGREE EXAMINATION – **MATHEMATICS**

SECOND SEMESTER – **NOVEMBER 2024**

PMT2MC02 – REAL ANALYSIS-II

Date: 09-11-2024 Dept. No. Max. : 100 Marks

Time: 01:00 pm-04:00 pm

	SECTION A – K1 (CO1)
	Answer ALL the questions **(5 x 1 = 5)**
1	**Answer the following**
a)	When is a linear operator defined on vector space X considered invertible?
b)	Find the outer measure of a finite set.
c)	Identify the positive part of the function $f(x)=x^2-1$ defined in $[-2,2]$.
d)	Let $X=[0,1]$ and define $f: X \to R$ by $f(x)=x^{-1/3}$, show that $f \in L^1(\mu)$ but $f \notin L^3(\mu)$.
e)	State Hahn decomposition theorem.
	SECTION A – K2 (CO1)
	Answer ALL the questions **(5 x 1 = 5)**
2	**Multiple Choice Questions**
a)	The directional derivative of the function $f=4xy^3-3x^2y^2z^2$ at the point $(2,-1,2)$ along z-axis is (i) 6 (ii) 20 (iii) $\frac{1}{6}$ (iv) – 48
b)	Which of the following is a σ-algebra on $X=\{1,2,3,4\}$ (i) $\{\varnothing, X\}$ (ii) $\{\varnothing,\{1\},\{2\},\{3\},\{4\},\{1,2,3,4\}\}$ (iii) $\{\varnothing,\{1\},\{1,3\},\{1,2,4\},\{1,2,3,4\}\}$ (iv) $\{\varnothing,\{1\},\{2,3,4\},\{1,2,3,4\}\}$.
c)	A real-valued function $\varphi(x)$ is simple if (i) $\varphi(x)$ is finite-valued. (ii) $\varphi(x)$ is non-negative and finite-valued. (iii) $\varphi(x)$ is finite-valued and measurable. (iv) $\varphi(x)$ is non-negative, taking only finite number of distinct values.
d)	A triple $[X,S,\mu]$ is called a measure space if (i) $[X,S]$ is a measurable space (ii) μ is a measure on S (iii) $[X,S]$ is a measurable space and μ is a measure on S (iv) None of the above.
e)	For measurable spaces $[X,S]$ and $[Y,T]$, $A\times B$ is a measurable rectangle if (i) $A\subseteq X, B\subseteq Y$ and $A\in S, B\in T$. (ii) $A\subseteq X, B\subseteq Y$ and $A\notin S, B\in T$. (iii) $A\subseteq X, B\subseteq Y$ and $A\in S, B\notin T$. (iv) $A\subseteq X, B\subseteq Y$ and $A\notin S, B\notin T$.
	SECTION B – K3 (CO2)
	Answer any THREE of the following **(3 x 10 = 30)**
3	Define $f: R^2 \to R^2$ by $f(x,y)=¿$). Then verify the following: (i) Directional derivative of *f* at (0, 0) exists. (ii) The function *f* is differentiable at (0, 0). (iii) Derivative of *f* at (0, 0) is invertible or not.
4	Justify the claim: All intervals are measurable.

5	Calculate the value of $\int_0^1 \frac{x^{1/3}}{1-x} \log\frac{1}{x}\, dx$.

6	Examine whether $\|f+g\|_p \leq \|f\|_p + \|g\|_p$ for $p \geq 1$ and $f, g \in L^p(\mu)$. Also, state the condition for equality and prove it.
7	For a signed measure, exhibit that a countable union of positive sets is a positive set.
	SECTION C – K4 (CO3)
	Answer any TWO of the following **(2 x 12.5 = 25)**
8	Determine if the class of Lebesgue measurable sets is a σ-algebra.
9	Let $\{f_n\}$ be a sequence of measurable functions such that $\|f_n\| \leq g$, where g is integrable and let $\lim f_n = f\ a.e.$ Then show that *f* is integrable and $\lim \int f_n dx = \int f dx$.
10	For a measure μ be a defined on a σ-ring $S$and class $\overline{S} = \{E\,N : for\ any\ sets\ E, N\ with\ E\ S\ while\ N\ is contained \in some\ set \in S\ of\ zero\ measure\}$,define the set function $\overline{\mu}$ by $\overline{\mu}(E \Delta N) = \mu(E)$. Prove that $\overline{S}$ is a σ-ring and $\overline{\mu}$ is a complete measure on $\overline{S}$.
11	A signed measure can be decomposed into difference of two measures and the decomposition is unique. Prove this assertion.
	SECTION D – K5 (CO4)
	Answer any ONE of the following **(1 x 15 = 15)**
12	If *f* is Riemann integrable and bounded over the finite interval [*a*, *b*], then verify whether *f* is integrable and $R\int_a^b f dx = \int_a^b f dx$. What do you say about converse of the statement? Justify your claim with an example.
13	Let $[X, S, \mu]$ and $[Y, T, v]$ be the σ-finite measure spaces. For $V \in S \times T$, denote $\varphi(x)\ as\ v(V_x)$, $\psi(y)\ as\ \mu(V^y)$, for each $x \in X, y \in Y$. Then demonstrate how φ is S-measurable, ψ is T-measurable and $\int_X \varphi\, d\mu = \int_Y \psi\, dv$.
	SECTION E – K6 (CO5)
	Answer any ONE of the following **(1 x 20 = 20)**
14	Establish the proof of the statement: Let f be a C'-mapping of an open set $E \subset R^{n+m}$ into R^n such that $f(a,b) = 0$ for a point $(a,b) \in E$ and $A = f'(a,b)$. Assume that A_x is invertible. Then there exist open sets $U \subset R^{n+m}$ and $W \subset R^m$, with $(a,b) \in U$ and $b \in W$, such that: (i) To every $y \in W$ there is a unique x such that $(x,y) \in U$ and $f(x,y) = 0$. (ii) If x is defined by $g(y)$, then g is a C'-mapping of W into R^n, $g(b) = a$, $f(g(y), y) = 0\ for\ y \in W$ and $g'(b) = -(A_x)^{-1} A_y$.
15	State Fatou's Lemma and validate it.

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